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Convergents to turbulence functions

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A method is described for constructing approximations, to statistical functions, that are uniformly convergent in time, starting with the expansion of the functions as Taylor series in time. The principal tool is a technique for expanding the Fourier transform of the unknown function by use of a set of orthonormal functions. Application to the Lagrangian velocity correlation and the eddy diffusivity for marked particles in a three-dimensional random velocity field yields results that agree excellently with computer simulations. The approximation procedure is extended to expansions in strength parameters (e.g. Reynolds number expansion) and to an expansion about the direct-interaction approximation. The latter is based on a new model representation of the direct-interaction approximation. An implication of the work is that the usual diagram expansions, obtained through term-by-term averaging over a Gaussian distribution, may not uniquely determine the functions they represent; it may be that truly meaningful expansions are possible, in general, only for distributions which bound the amplitudes in the individual realizations.

1. Introduction

Suppose that an initial ensemble of turbulent velocity fields is specified by all its moments. Then straightforward manipulations of the equations of motion yield formal expansions for any desired statistical functions in the form of Taylor series in time or in the form of power series in a strength parameter associated with the non-linear terms in the equations of motion (Batchelor 1953; Kraichnan 1966). In addition, several less elementary formal expansion schemes for turbulence have been proposed in recent years, based either on rearrangements of the Taylor and strength-parameter expansions (Wyld 1961; Kraichnan 1961; Lee 1965; Orszag 1966) or on expansion about an assumed approximate statistical state (Edwards 1964; Herring 1965, 1966; Edwards & McComb 1969; Phythian 1969).

In an assessment of the formal expansion schemes made several years ago (Kraichnan 1966), the present author pointed out that any of them very likely has a zero radius of convergence in the relevant parameter, while, even if the radius of convergence were infinite, impractically many terms might have to be summed to get valid approximations to statistical functions from truncations of the series. Moreover, the complete expansions contain insufficient information to determine the statistical functions uniquely in the absence of additional sources

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of information about analyticity properties. For example, the sum of a Taylor series is determinate only up to a function whose Taylor series vanishes.

The present paper returns to these questions and reports some techniques for constructing uniformly convergent approximations to statistical functions by combining qualitative information with the quantitative data provided by the coefficients of formal expansions. The principal tool is a way of expanding the Fourier transform of the unknown function into contributions from a suitably chosen complete set of continuous orthogonal functions. The first part of the paper develops the mathematical methods and illustrates them by application to the series expansion of some simple functions. Application is then made to the Taylor series expansion in time for the Lagrangian velocity correlation of a particle moving in a random, isotropic, incompressible velocity field, and the results are compared with computer experiments. Finally, more general applications, and modifications, are discussed, including application to strength-parameter (i.e. Reynolds or Péclet number) expansions and to an expansion about the direct-interaction approximation. The latter expansion is a modification of one recently proposed by Phythian (1969).

The general implication of the paper is that, first, it is indeed possible to obtain converging sequences of approximations to turbulence functions by suitable manipulation of known formal expansions and that, secondly, surprisingly good quantitative approximations can be obtained, at least in some cases, by using only the lowest few coefficients in the expansions. The techniques proposed herein are, by their own nature, non-unique, and considerable extension of the convergence theory seems feasible. It is therefore hopeful that modifications and alternatives will yield better and more widely applicable approximation methods in the future. Padé approximants, another technique for handling divergent series, have already shown some promise for turbulence applications (Kraichnan 1968, 1970*a*).

2. Approximation of power series by orthogonal expansion of the Fourier transform

Suppose that $f(t)$ and $\rho(a)$ are even functions related by

$$f(t) = \int_{-\infty}^{\infty} \rho(a) \cos(at)\, da. \tag{2.1}$$

Using the Taylor series for cos, we have

$$f(t) = \sum_{n=0}^{\infty} (-1)^n c_{2n} t^{2n}/2n!, \tag{2.2}$$

where

$$c_{2n} = \int_{-\infty}^{\infty} \rho(a)\, a^{2n}\, da. \tag{2.3}$$

Thus the problem of reconstructing $f(t)$ from its Taylor series is equivalent to that of finding $\rho(a)$ from its moments. We seek to do this by expanding $\rho(a)$ in the form

$$\rho(a) = \sum_{n=0}^{\infty} b_{2n} P_{2n}(a)\, w(a), \tag{2.4}$$

where the $P_n(a)$ are the set of polynomials orthogonal in $(-\infty, \infty)$ with respect to some even, everywhere-positive weight $w(a)$:

$$P_n(a) = \sum_{m=0}^{n} d_{nm} a^m \quad (d_{nn} > 0), \quad \int_{-\infty}^{\infty} P_n(a) P_m(a) w(a) \, da = \delta_{nm}. \tag{2.5}$$

From (2.4) and (2.5), we have

$$b_{2n} = \int_{-\infty}^{\infty} \rho(a) P_{2n}(a) \, da = \sum_{m=0}^{n} d_{2n, 2m} c_{2m}, \tag{2.6}$$

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} b_{2n} \int_{-\infty}^{\infty} P_{2n}(a) \cos(at) \, w(a) \, da \\ &= \sum_{n=0}^{\infty} b_{2n} P_{2n}\left(i \frac{d}{dt}\right) \zeta(t), \end{aligned} \tag{2.7}$$

where $\zeta(t)$ is the cosine transform of $w(a)$. Each coefficient b_{2n} in (2.7) is determined by the c_{2m} $(m \leqslant n)$. Thus the successive terms of (2.7) can be determined from those of the original Taylor series (2.2). We must now justify the formal manipulations and find conditions under which (2.7) converges.

First of all, normalizability of the $P_n(a)$ requires that $w(a)$ fall off faster than algebraically as $|a| \to \infty$. On the other hand, if we expand $\rho(a)/[w(a)]^{\frac{1}{2}}$ in the orthonormal functions $P_n(a)\,[w(a)]^{\frac{1}{2}}$, Bessel's inequality yields

$$\sum_{n=0}^{\infty} |b_{2n}|^2 \leqslant \int_{-\infty}^{\infty} \frac{[\rho(a)]^2}{w(a)} \, da, \tag{2.8}$$

wherein the integral exists if $\rho(a)$ is in L^2 and $w(a)$ falls off sufficiently slowly at infinity. If $[w(a)]^{\frac{1}{2}} \cos(at)$ is expanded in the $P_n(a)\,[w(a)]^{\frac{1}{2}}$, Bessel's inequality yields

$$\sum_{n=0}^{\infty} \left| \int_{-\infty}^{\infty} P_{2n}(a) \cos(at) \, w(a) \, da \right|^2 \leqslant \int_{-\infty}^{\infty} w(a) \cos^2(at) \, da, \tag{2.9}$$

where the right-hand side exists in consequence of the already required integrability of $w(a)$. It now follows, from applying Schwarz's inequality to (2.8) and (2.9), that (2.7) converges whenever the right-hand side of (2.8) exists.

Let $\rho_r(a)$ and $f_r(t)$ represent the respective truncations of (2.4) and (2.7) to $n \leqslant r$. Then

$$f_r(t) = \int_{-\infty}^{\infty} \rho_r(a) \cos(at) \, da, \tag{2.10}$$

where the interchange of integration and summation is permitted at any r because the integrals converge separately for each n. Subtracting (2.10) from (2.1), multiplying the integrand in the result by $[w(a)/w(a)]^{\frac{1}{2}}$, and again using Schwarz's inequality, we have

$$|f(t) - f_r(t)|^2 \leqslant \int_{-\infty}^{\infty} \frac{|\rho(a) - \rho_r(a)|^2 \, da}{w(a)} \int_{-\infty}^{\infty} w(a) \cos^2(at) \, da. \tag{2.11}$$

The second integral on the right-hand side of (2.11) has a bound independent of t, while the first integral approaches zero as $r \to \infty$, if the right-hand side of (2.8) exists and if the $P_{2n}(a)\,[w(a)]^{\frac{1}{2}}$ are a complete orthonormal set of even functions. Under these conditions, $f_r(t) \to f(t)$, uniformly, as $r \to \infty$.

Let us now consider the significance of these sufficient conditions for uniform convergence of (2.7) to $f(t)$. Existence of the right-hand side of (2.8) is assured, for some $w(a)$ with normalizable $P_{2n}(a)$, provided that $\rho(a)$ is in L^2 and falls off faster than algebraically at infinity. (The first-named condition is not always necessary for convergence of (2.7), provided that (2.9) converges fast enough; this will be illustrated by an example in §3.) Faster-than-algebraic fall-off implies that $f(t)$ and all its derivatives are continuous everywhere. The appearance of this condition expresses the fact that $f(t)$ cannot be continued through a singularity at $t \neq 0$ if the only information available is its behaviour about $t = 0$.

A sufficient condition for completeness of the $P_{2n}(a)\,[w(a)]^{\frac{1}{2}}$ follows from the fact that there is no even $\rho(a)$, other than $\rho(a) \equiv 0$, such that the c_{2n} all vanish and $\int_{-\infty}^{\infty} |\rho(a)| \cosh(qa)\,da$ exists, where q is a non-zero, positive constant. The following concise proof of the latter theorem is due to Orszag (private communication). Set $f(t) = \int_{-\infty}^{\infty} \rho(a)\,e^{iat}\,da$. Then $f(t)$ is analytic in the strip $-q < \operatorname{Im} t < q$, and $f^{(n)}(0) = i^n \int_{-\infty}^{\infty} a^n \rho(a)\,da = 0$. Hence, $f(t) = 0$ for $|t| < q$ and, by analytic continuation, $f(t) = 0$ throughout the strip. Therefore $\rho(a) \equiv 0$, by the Fourier inversion formula. Now suppose that $w(a)$ is $O(e^{-q|a|})$ as $|a| \to \infty$ and that the $P_{2n}(a)\,[w(a)]^{\frac{1}{2}}$ were incomplete with respect to some even $\rho(a)$ for which the right-hand side of (2.8) exists. We assume that the even function $w(a)$ is positive, non-zero everywhere, and bounded. Incompleteness would mean that $\rho(a) - \rho_\infty(a)$, whose moments all are zero, would be non-zero in mean square. But it follows from Bessel's inequality, and the properties of $w(a)$, that

$$\int_{-\infty}^{\infty} |\rho(a) - \rho_\infty(a)|^2 \cosh(qa)\,da$$

exists and therefore that $\int_{-\infty}^{\infty} |\rho(a) - \rho_\infty(a)| \cosh(q'a)\,da$ exists, if $0 < q' < \frac{1}{2}q$. Hence, the preceding theorem would be violated. It follows that the $P_{2n}(a)\,[w(a)]^{\frac{1}{2}}$ are complete with respect to $\rho(a)$ for which (2.8) exists, and that (2.7) converges uniformly to $f(t)$ for $-\infty \leqslant t \leqslant \infty$, provided that $w(a)$ is everywhere positive, non-zero, and bounded and that $w(a) = O(e^{-q|a|})$ as $|a| \to \infty$, for some positive q.

Exponential fall-off of $\rho(a)$ as $|a| \to \infty$ implies that (2.2) has a finite, non-zero radius of convergence. The converse is not always true, and analyticity of $f(t)$ at $t = 0$ does not ensure that $w(a)$ exists such that (2.7) converges uniformly to $f(t)$. The following three statements express sufficient conditions for convergence. In all three statements, we assume that $w(a)$ is even, everywhere positive, non-zero, and bounded, and that

$$e^{-q''|a|} < w(a) < e^{-q'|a|}, \quad \text{as} \quad |a| \to \infty, \quad \text{where} \quad 0 < q' < q''.$$

(1) Equation (2.7) converges uniformly to $f(t)$ if $\int_{-\infty}^{\infty} [f(t)]^2 dt < C$ and $\int_{-\infty}^{\infty} [d^n f(t)/dt^n]^2 dt < 2n!\,q^{-2n} C$ $(n = 1, 2, \ldots)$, for some positive C and q, and $q'' < q$.

(2) Equation (2.7) converges uniformly to $f(t)$ if $\int_{-\infty}^{\infty}[f(t)]^2dt$ exists, and $f(t)$ is analytic at $t = 0$ with radius of convergence q for (2.2), and, for $|a| >$ some α, $\rho(a)$ is non-negative with a bound independent of a, and $q'' < q$.

(3) Equation (2.7) converges uniformly to $f(t)$ if $f(t)$ is analytic in the strip $t = u+iv$, $|v| < q$ and $\int_{-\infty}^{\infty}|f(u+iv)|\,du$ is uniformly bounded within this strip, and $f\to 0$ uniformly within this strip as $|u|\to\infty$ and $q'' < 2q$.

Statement (1) implies
$$\int_{-\infty}^{\infty}(1+t^2)^{-1}\,|f(t)|\,dt < \infty.$$

Then differentiation of the Fourier transform relations under the integral signs can be justified by appeal to 'generalized functions', and Parseval's formula is valid for $f(t)$ and its derivatives in any case where either of the integrals in the formula exists (Lighthill 1958). The given bounds then yield
$$2\pi\int_{-\infty}^{\infty}a^{2n}|\rho(a)|^2da < 2n!\,q^{-2n}C \quad (n = 0,1,2,\ldots).$$

Hence,
$$\sum_{n=0}^{\infty}(q'')^{2n}\,(2n!)^{-1}\int_{-\infty}^{\infty}a^{2n}\,|\rho(a)|^2da$$

exists, and, interchanging integration and summation by the monotone convergence theorem, we have that
$$\int_{-\infty}^{\infty}|\rho(a)|^2\cosh(q''a)\,da$$
and, hence, (2.8) exist.

In statement (2), $f^{(2n)}(0) < 2n!\,q^{-2n}C$ (for some $C > 0$),
$$\int_{-\infty}^{\infty}|\rho(a)|^2da < \infty, \quad \text{and} \quad \int_{-\infty}^{\infty}|\rho(a)|\,da < \infty,$$

the last since $f(0)$ exists and $\rho(a) \geqslant 0\,(|a| > \alpha)$. Equation (2.1) and the definition of derivative give
$$f^{(2n)}(0) = \lim_{\tau\to 0}\int_{-\infty}^{\infty}a^{2n}\rho(a)\frac{\sin^{2n}(\tau a)}{(\tau a)^{2n}}\,da,$$

whence
$$\int_{-\infty}^{\infty}a^{2n}\rho(a)\,da = f^{2n}(0), \quad \text{by} \quad \rho(a)\geqslant 0 \quad (|a| > \alpha)$$

and the properties of $\sin^{2n}(\tau a)/(\tau a)^{2n}$. The bounds on $f^{2n}(0)$, $\rho(a)\geqslant 0\,(|a| > \alpha)$, and
$$\int_{-\infty}^{\infty}|\rho(a)|\,da < \infty \quad \text{then give} \quad \int_{-\infty}^{\infty}|\rho(a)|\cosh(q''a)\,da < \infty,$$

whence, since $\rho(a)$ is bounded for $|a| > \alpha$, $\int_{-\infty}^{\infty}|\rho(a)|^2\cosh(q''a)\,da$ exists, so that (2.8) exists.

To prove statement (3), we write

$$\rho(a) = (2\pi)^{-1}\int_{-\infty}^{\infty} f(t)\, e^{-iat}\, dt$$

and (for $a > 0$) shift the integration contour from the real t axis to the line $t = u - iq + i\epsilon$, where ϵ is an arbitrarily small positive constant, returning to the t axis at $|t| = \infty$. The uniform bound on the integral of f then shows that the part of the contour parallel to the real axis makes a contribution $O(e^{-(q-\epsilon)a})$ to $\rho(a)$, while the condition that $f \to 0$ uniformly shows that the returns make a vanishing contribution. Also, the uniformity, analyticity, and absolute integrability of $f(t)$ on the real axis imply that

$$\int_{-\infty}^{\infty} [f(t)]^2 dt \quad \text{and, hence,} \quad \int_{-\infty}^{\infty} [\rho(a)]^2 da$$

exist. It follows that (2.8) exists.

The most inclusive of the three statements is (1), since the bounds on integrals given therein are necessary conditions for (2.8) to converge with some $w(a)$ that yields a complete set of orthonormal functions. However, as an example in § 3 will show, (2.7) can converge even in some cases where (2.8) does not.

In statement (2), the boundedness of $\rho(a)$ at large a is a stronger condition than needed, and probably can simply be omitted for physical applications. In fact, if

$$\int_{-\infty}^{\infty} \rho(a) \cosh(q''a)\, da \quad \text{and} \quad \int_{-\infty}^{\infty} |\rho(a)|^2 da$$

both exist, with $\rho(a)$ non-negative for $|a| > \alpha$, the only condition under which

$$\int_{-\infty}^{\infty} |\rho(a)|^2 \cosh(2q''a)\, da$$

could fail to exist would appear to be if $\rho(a)$ exhibited a sequence of peaks of exponentially increasing sharpness, as $|a| \to \infty$. An example is given below.

A corollary of statement (2) is that a non-negative, even $\rho(a)$ is uniquely determined by its moments c_{2n} $(n = 0, 1, 2, \ldots)$ if the latter are bounded by $c_{2n} < 2n!\, q^{-2n} C$. This follows from the fact that the transform $f(t)$, which is analytic in the strip $|\mathrm{Im}\, t| < q''$, is uniquely determined. The uniqueness of $\rho(a)$ is a variant of Carleman's theorem (Wall 1948), whose usual statement says that an even, non-negative, integrable $\rho(a)$ is uniquely determined by the c_{2n} if $\sum_{n=0}^{\infty} (1/c_{2n})^{1/2n}$ diverges.

The distinctions among the three statements can be elucidated by some simple examples. The choice $f(t) = \exp(-t^2)$ satisfies the conditions of all three statements. The choice $f(t) = t^{-1}\sin(t)$ satisfies statements (1) and (2), but not (3). The choice $f(t) = [1 + (t-1)^2]^{-1} + [1 + (t+1)^2]^{-1}$ satisfies (1) and (3), but not (2) $[\rho(a) \propto e^{-|a|} \cos a]$. The sum of the last two examples satisfies only (1).

Now for some examples that satisfy none of the statements, and for which there is no $w(a)$ with complete orthonormal $P_{2n}(a)\,[w(a)]^{\frac{1}{2}}$ such that (2.8) exists.

The function $\cos(t^2)$ is analytic everywhere, but its transform oscillates with constant amplitude and ever-increasing frequency as $|a|$ increases;

$$(1+t^2)^{-6}\cos(t^2)$$

also violates the conditions of all three statements. The function $f(t) = (1-t^2)^{\frac{1}{2}}$ for $t^2 < 1$, $= 0$ for $t^2 \geqslant 1$, is analytic at $t = 0$, but its transform oscillates and goes off as a power for large $|a|$. Finally, consider $\rho(a) = 0$ everywhere except $\rho(a) = n^{-1}e^n$ for $||a|-n| < e^{-2n}$ $(n = 1, 2, \ldots, \infty)$. Here $f(t)$ is bounded and, for all real t, $|d^nf(t)/dt^n| < n!C$, for some positive C, so that $f(t)$ is analytic in a strip of half-width one centred on the real axis. But, although

$$\int_{-\infty}^{\infty} |\rho(a)| \cosh(q''a)\,da \quad \text{exists for} \quad 0 \leqslant q'' < 1,$$

$$\int_{-\infty}^{\infty} |\rho(a)|^2 \cosh(q''a)\,da \quad \text{exists only for} \quad q'' = 0.$$

This example spectacularly violates the boundedness condition of statement (2), and it violates the integrability conditions of statements (3) and (1).

If $\rho(a)$ is reasonably well-behaved at large $|a|$, the following modification of statement (1) is relevant.

(1′) Equation (2.7) converges uniformly to $f(t)$ if $\int_{-\infty}^{\infty}[f(t)]^2dt$ exists and $|f(t)| < C$, $|d^nf(t)/dt^n| < n!q^{-n}C$ $(n = 1, 2, \ldots)$, for some positive C and q independent of t, provided that $q'' < 2q$ and that there exist positive α, b, δ, and D such that

$$|\rho(a)| < D\left|\int_{-1}^{1}(1-s^2)\,\rho(a'+s\delta)\,ds\right| \quad \text{for all} \quad |a| > \alpha$$

and for some $|a'-a| < b$. (The final condition excludes infinities and oscillations of unbounded rapidity at large $|a|$.)

To prove (1′), we first note that the bounds on f and its derivatives imply that f is analytic and bounded in a strip of half-width q centred on the real t axis. Consider the function $f_\nu(t) = f(t)\exp(-\nu t^2)$, where $\nu > 0$. Its transform $\rho_\nu(a)$ can be estimated by the same contour shift used in proving statement (3). Again, the returns give a vanishing contribution, while the horizontal part of the contour gives a contribution of the form $e^{-(q-\epsilon)a}$ times an absolutely convergent integral over u. Now integrate this contribution between $a'-\delta$ and $a'+\delta$, with the weighting $1-s^2$ $(a = a'+s\delta)$, and exchange the order of integrations. The resulting integral over u is absolutely convergent and independent of ν as $\nu \to 0$. It then follows from the stated condition on $\rho(a)$ that $|\rho(a)| = O(e^{-(q-\epsilon)a})$ as $a \to \infty$. From this, and Parseval's relation for $f(t)$, it follows that (2.8) exists for $q'' < 2q$.

It is essential to remember that $f(t)$ is never uniquely determined by knowledge of the c_{2n} alone, whatever the values of the latter. Some sort of global information about $f(t)$ or $\rho(a)$ is needed also. Suppose that $f(t)$ satisfies conditions for convergence of (2.7). Then $f(t)+g(t)$, where $g(t) = |t|^{-\frac{3}{2}}\exp(-|t+t^{-1}|)$, has precisely the same c_{2n} as $f(t)$, and $g(t)$ is continuous, along with all its derivatives, at all real t. However, the cosine transform of $g(t)$ is

$$\pi^{-\frac{1}{2}}\exp\left[-2(1+a^2)^{\frac{1}{4}}\cos\left(\frac{1}{2(1+a^2)^{\frac{1}{2}}}\right)\right]\cos\left[(1+a^2)^{\frac{1}{4}}\sin\left(\frac{a}{2(1+a^2)^{\frac{1}{2}}}\right)\right]$$

(Roberts & Kaufman 1966), which falls off at large $|a|$ like $\exp(-2|a|^{\frac{1}{2}})$ and thereby fails to satisfy our conditions for uniform convergence. A similar, but more general, example of a $g(t)$ which is continuous, along with all its derivatives, for all real t, but which has a vanishing Taylor expansion about $t = 0$, is the cosine transform of

$$\exp(-|a|^s)\cos[|a|^s\tan(\pi s/2)] \quad (0 < s < 1)$$

(Shohat & Tamarkin 1943).

Convergence of the $\rho_r(a)$ to $\rho(a)$ in mean square implies that $\rho_r(a)$ converges to $\rho(a)$ at all but a zero-measure set of points a. If there are no discontinuities in $\rho(a)$ or $w(a)$, at which Gibbs phenomena would be expected, we anticipate that, when the completeness condition is satisfied, $\rho_r(a)$ converges to $\rho(a)$ at every finite a. (However, the author has found no proof of this, in the literature, that is valid for the general weights $w(a)$ which we admit.) In particular, we anticipate that $\pi\rho_r(0)$ converges to $\int_0^\infty f(t)\,dt$ if neither $\rho(a)$ nor $w(a)$ are discontinuous at $a = 0$. It should be noted here that this last convergence property does not follow from the uniform convergence of $f_r(t)$ to $f(t)$.

If $f(t)$ and $\rho(a)$ are odd, instead of even, (2.1) is replaced by

$$f(t) = \int_{-\infty}^{\infty} \rho(a)\sin(at)\,da, \tag{2.12}$$

and the analysis goes through with the obvious change that the expansion is now in odd $P_n(a)$. The convergence and completeness criteria are the same as in the even case. Since the procedures are linear, the same convergence and completeness criteria apply to $f(t)$ which are the sums of an odd and an even part.

A number of modifications and generalizations of the orthogonal expansion method are possible. Simple transformations extend the applicability to some functions $f(t)$ which are excluded by the convergence conditions developed above. For example, if $f(t)$ is odd and approaches a non-zero limit as $t \to \infty$, its sine transform is unbounded. In this case we can apply the expansion to the even function $df(t)/dt$ and seek convergents to $f(t)$ by integrating the convergents to its derivative. To take another case, suppose that $f(t)$ has a branch point at some $t_1 \neq 0$. We can transform to a new variable s so as to map the singularity-free region about $t = 0$ upon the entire real s-axis and determine the power-series expansion in s from that in t. The expansion procedure can then be carried out in terms of s, and the final results transformed back to yield convergents to $f(t)$ between $t = 0$ and $t = t_1$. Finally (2.1) can be replaced by other integral transforms whose kernels have known power-series expansions, the use of the Stieltjes transform (Kraichnan 1970a) being of particular interest.

If $\int_{-\infty}^{\infty} w(a)\,da$ is a fixed constraint, the right-hand side of (2.8) is stationary to variations, and has a minimum, if $[w(a)]^2 \propto [\rho(a)]^2$ for all a. If the characteristic width of $w(a)$ is much smaller than that of $\rho(a)$, the integrand is large at large a, and the reverse inequality makes the integrand large at small a. This suggests that it should be possible to accelerate convergence of the orthogonal expansion method by incorporating adjustable parameters in $w(a)$ and, with suitable precautions, choosing them, at each r above some minimum, to minimize $\sum_{n=0}^{r} |b_{2n}|^2$ for the rth convergent. We shall illustrate an heuristic scheme of this kind in § 3.

A similar variational procedure provides an alternative, to the differentiation method mentioned above, for finding convergents to an $f(t)$ which has a nonzero limit as $t \to \infty$. We apply the orthogonal expansion technique to $f(t)+K$, and determine K, at each level of the expansion, so as to minimize $\sum_{n=0}^{r} |b_{2n}|^2$.

The orthogonal expansion method can be modified to give uniformly convergent approximations to $f(t)$ for all real t when the given information is the values at a discrete set of points in the interval $0 \leqslant t \leqslant t_1$, rather than the derivatives at $t = 0$. Suppose these points are 0, t_1/N, $2t_1/N$, $3t_1/N$, ..., t_1. Then we consider the infinite sequence β_n $(n = 0, 1, 2, \ldots)$, given by 0, $1/N$, $2/N$, $3/N$, ..., 1, $\frac{1}{2}N$, $\frac{3}{2}N$, $\frac{5}{2}N$, ..., $(2N-1)/2N$, $\frac{1}{4}N$, $\frac{3}{4}N$, ..., $(4N-1)/4N$, ... An orthonormal set of functions is now constructed by linear combination of the functions $[w(a)]^{\frac{1}{2}} \cos(\beta_n t_1 a)$, in such a way that the nth orthonormal function involves only β_s for $s \leqslant n$. The weight $w(a)$ satisfies the conditions previously imposed, and we can ensure normalizability of the present orthogonal set by requiring, in addition, that $w(a)$ be a monotone decreasing function of $|a|$. The coefficients of the expansion of $\rho(a)$ by means of this orthonormal set are determined by the values of $f(t)$ at the points $\beta_n t_1$ in direct analogy to the preceding analysis. The sufficient conditions for uniform convergence for all real t carry over, the proofs now depending on the fact that an analytic function which is zero at the infinitesimally spaced set of points $\beta_n t_1$ must vanish identically. The original $N+1$ data points thus provide a set of approximants that are embedded in a uniformly convergent sequence. In contrast to (2.7), the present procedure gives $f(t)$ as a linear combination of $\zeta(t)$ and the functions $\zeta(t \pm \beta_n t_1)$.

3. Illustrations of the expansion method

For a first example, let

$$f(t) = \exp(-\tfrac{1}{2}t^2), \quad \rho(a) = (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}a^2), \quad w(a) = (4\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{4}a^2).$$

The $P_n(a)$ are the (suitably normalized) Hermite polynomials. Using these functions in (2.2)–(2.6), we find for the $|b_{2n}|^2$ the values 1, 0·125, 0·023, 0·005, 0·001, 0·0002, 0·00006, ... for $n = 0, 1, 2, \ldots$. This corroborates the expected convergence of (2.8). Figure 1 compares $\rho(a)$ with the $\rho_r(a)$ for $r = 0, 1, 2$. The values of $\rho_r(0)/\rho(0)$ are 0·707, 0·884, 0·950, 0·978, 0·990, 0·995, 0·998, ... for $r = 0, 1, 2, \ldots$, which indicates a rapid convergence to the exact value of $\int_0^\infty f(t)\,dt$.

If $w(a)$ is Gaussian, as in this example, the approximants have a very simple expression in the t domain. The cosine transform of $P_n(a)w(a)$ is proportional to $t^n \exp(-t^2)$, to use the present Gaussian, and it is easy to see that

$$f_r(t) = \exp(-t^2)\,[f(t)\exp(t^2)]_{(r)}, \tag{3.1}$$

where $[\;]_{(r)}$ denotes the truncation, at term t^{2r}, of the Taylor series of the function within $[\;]$. The latter, of course, is $\exp(\frac{1}{2}t^2)$ in the present case. Equation (3.1) displays clearly the uniform nature of the convergents $f_r(t)$ as compared to the simple truncations $[\exp(-\frac{1}{2}t^2)]_{(r)}$ of the original power series. Any of the latter

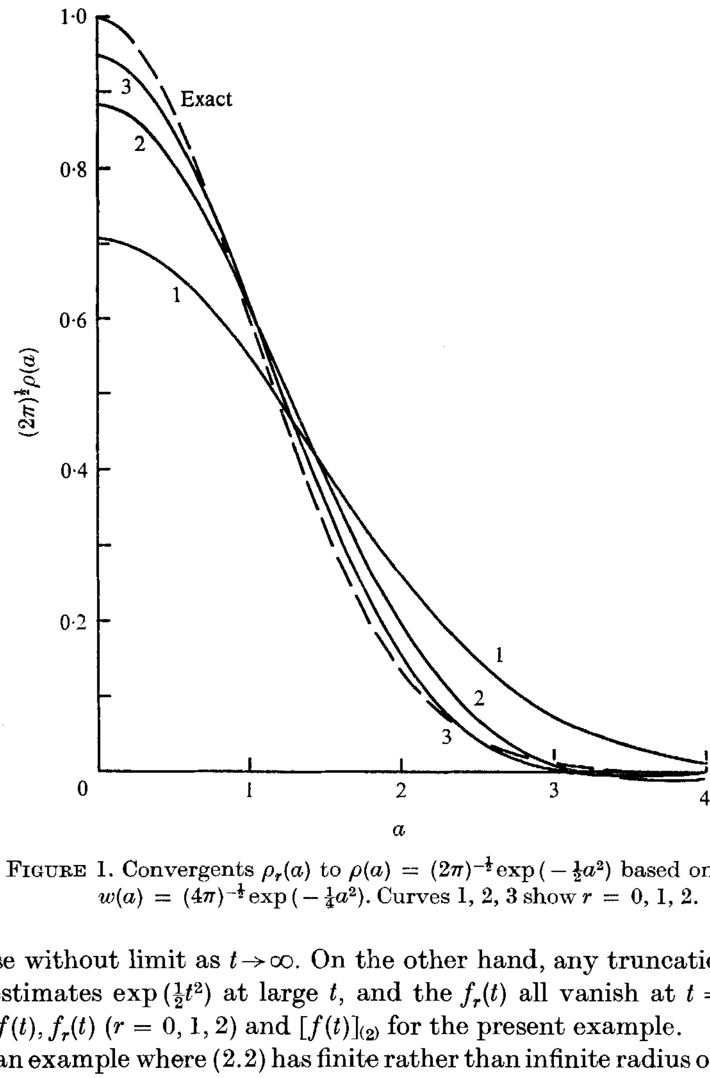

FIGURE 1. Convergents $\rho_r(a)$ to $\rho(a) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}a^2)$ based on weight $w(a) = (4\pi)^{-\frac{1}{2}}\exp(-\frac{1}{4}a^2)$. Curves 1, 2, 3 show $r = 0, 1, 2$.

increase without limit as $t \to \infty$. On the other hand, any truncation $[\exp(\frac{1}{2}t^2)]_{(r)}$ underestimates $\exp(\frac{1}{2}t^2)$ at large t, and the $f_r(t)$ all vanish at $t = \infty$. Figure 2 shows $f(t)$, $f_r(t)$ $(r = 0, 1, 2)$ and $[f(t)]_{(2)}$ for the present example.

For an example where (2.2) has finite rather than infinite radius of convergence, let us take $f(t) = \operatorname{sech}^2 t$ and take for $w(a)$ the cosine transform of sech t. (The positivity of $w(a)$ then follows immediately from the representation of sech as a limiting case of the elliptic function cn.) We can evaluate the $P_{2n}(a)$ and the b_{2n}, and find the $f_r(t)$ from (2.7) all without explicitly evaluating $w(a)$. We need only the moments of $w(a)$, which follow immediately from the Taylor series for sech t.

Figure 3 shows the resulting approximants $f_r(t)$ $(r = 0, 1, 2)$ compared with $f(t)$. The $|b_{2n}|^2$ are 1, 0·25, 0·0156, 0·0041, ... $(n = 0, 1, \ldots)$, and the values of $\rho_r(0)/\rho(0)$ are 1·57, 1·178, 1·104, 1·072, ... $(r = 0, 1, \ldots)$.

In both the preceding examples, $w(a)$ was chosen to qualitatively resemble $\rho(a)$, and clearly this is helpful for rapid convergence of the expansion method. The sufficient conditions for convergence found in § 2 say nothing about how fast the convergence is. Suppose we take $f(t) = \cos t$, and again take $w(a)$ to be

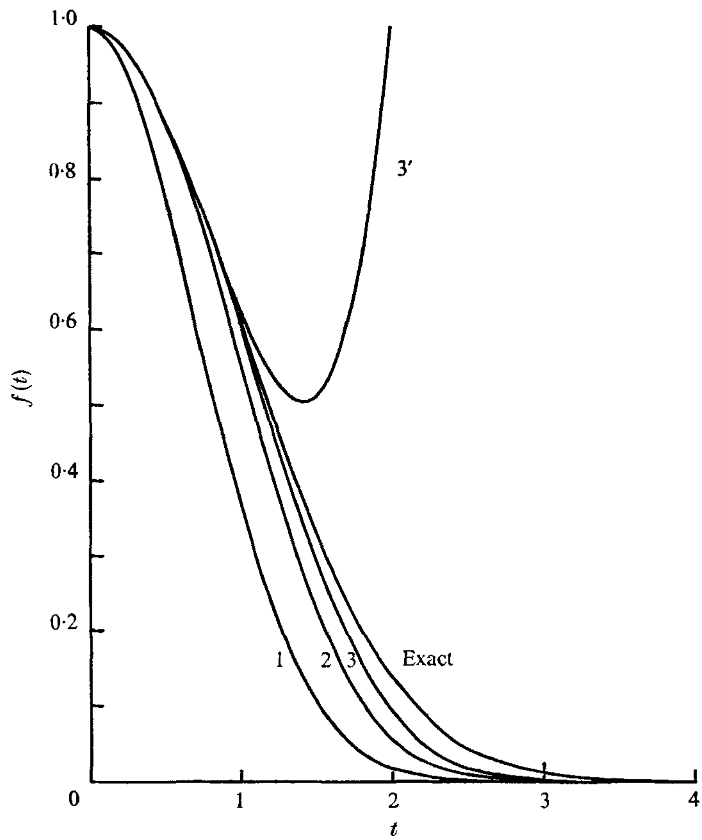

FIGURE 2. Convergents to $f(t) = \exp(-\frac{1}{2}t^2)$ from the $\rho_r(a)$ of figure 1. Curves 1, 2, 3 denote $r = 0, 1, 2$. Curve 3′ is $1 - \frac{1}{2}t^2 + \frac{1}{8}t^4$.

the transform of $\exp(-t^2)$, so that (3.1) holds. Here (2.8) diverges, since $\rho(a) = \frac{1}{2}[\delta(a-1) + \delta(a+1)]$ is not square-integrable. However, this is one of the cases, mentioned in § 2, where (2.9) converges strongly enough that the $f_r(t)$ converge anyhow. The successive $\rho_r(a)$ peak more and more strongly about $|a| = 1$, and the successive $f_r(t)$ follow $\cos t$ more and more faithfully until each dies to zero as $t \to \infty$. For large t, the $f_r(t)$ have, again, the advantage, over simple truncations of the Taylor series for $\cos t$, that they are bounded. But they are much less accurate than the Taylor-series truncations for t small enough that the latter are good approximations.

We wish next to give an example, not as yet supported by convergence theory, which suggests that the successive c_{2n} can be used to improve an initial choice of $w(a)$ and accelerate convergence. Again, take $f(t) = \text{sech}^2 t$, but now start with Gaussian $w(a)$. We know that any given Gaussian $w(a)$ will give divergence of (2.8), and, in this case, it can be verified that the $f_r(t)$ diverge also. However, suppose that, for $r = 1, 3, 5, \ldots$, we take a sum of Gaussians,

$$w(a) = \sum_{i=1}^{\frac{1}{2}(r+1)} A_i(4\pi)^{-\frac{1}{2}}\beta_i^{-1}\exp(-a^2/4\beta_i^2) \quad (A_i, \beta_i > 0), \tag{3.2}$$

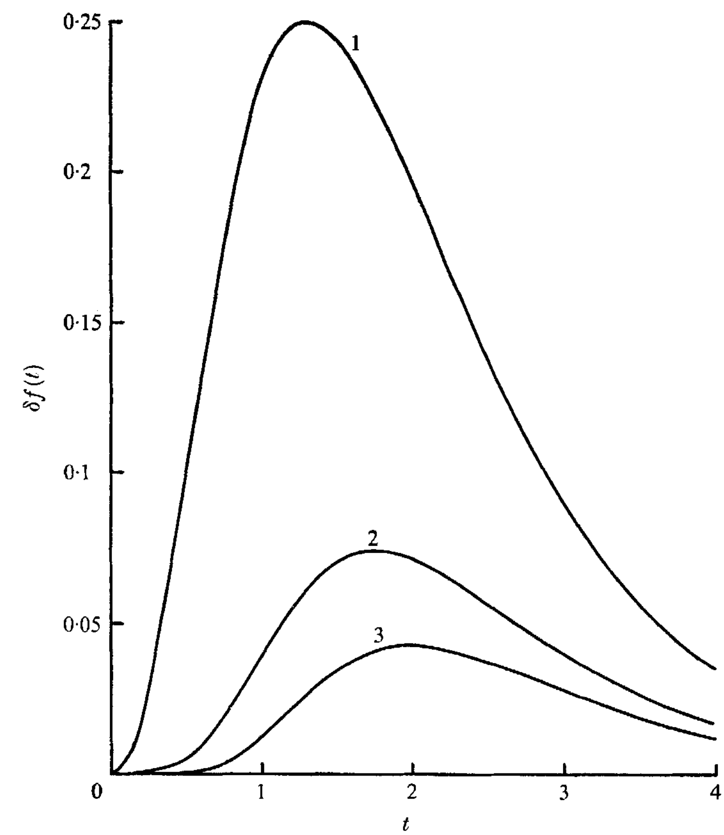

FIGURE 3. Errors $\delta f_r(t) = f_r(t) - f(t)$ in convergents to $f(t) = \text{sech}^2 t$ formed with the transform of sech t as weight. Curves 1, 2, 3 show $r = 0, 1, 2$.

where the $r+1$ parameters A_i and β_i are adjusted so as to minimize $\sum_{n=0}^{r} |b_{2n}|^2$ and, thereby, to minimize projection on the higher orthogonal functions. For even r, we use the $w(a)$ formed for $r-1$. In the case of $f(t) = \text{sech}^2 t$, we find that, through $r = 5$, at least, the minimum value of the projection is zero. That is, the Taylor series for $\text{sech}^2 t$ can be matched exactly, through order $2r$, by

$$f_r(t) = \sum_{i=1}^{\frac{1}{2}(r+1)} A_i \exp(-\beta_i^2 t^2) \quad (r \text{ odd}). \tag{3.3}$$

The values found for A_i, β_i and $\rho_r(0)/\rho(0)$ are as follows:

r	$\rho_r(0)/\rho(0)$	A_1	β_1	A_2	β_2	A_3	β_3
1	0·88623	1	1	—	—	—	—
3	0·96422	0·7847	0·8352	0·2153	1·4500	—	—
5	0·98494	0·5792	0·7469	0·4001	1·2327	0·0207	1·8234

Figure 4 shows the errors $\delta f_r(t) = f_r(t) - f(t)$. Note the difference in vertical scale between figures 3 and 4.

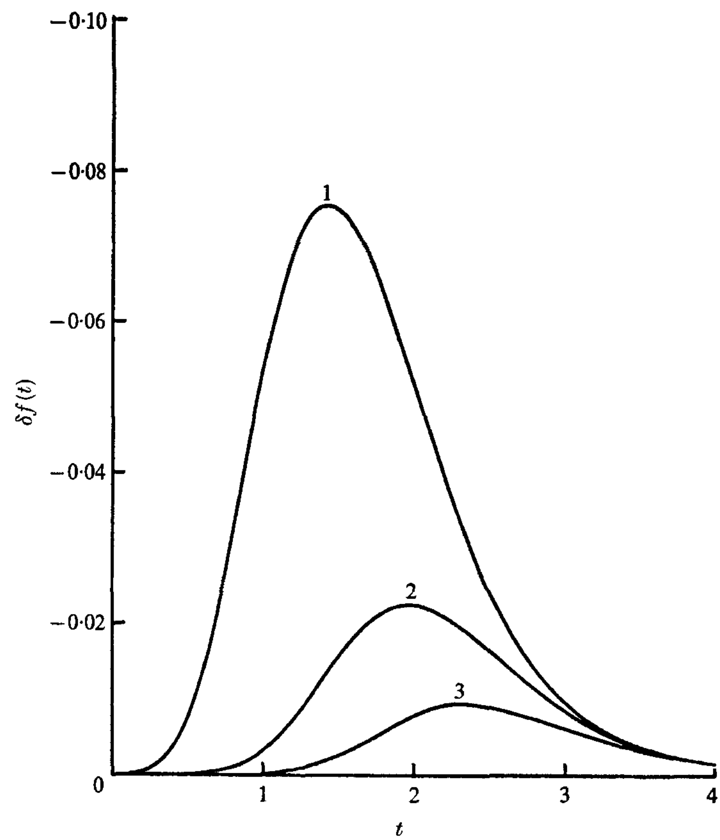

FIGURE 4. Errors $\delta f_r(t)$ for $f(t) = \text{sech}^2 t$ and weight taken as best-fitting sum of Gaussians (3.2). Curves 1, 2, 3 show $r = 1, 3, 5$.

As a final example, which goes even farther beyond the range of proved convergence, let us take

$$f(t) = 8\int_0^\infty p^3 \exp(-2p^2)\,\text{sech}^2(pt)\,dp. \tag{3.4}$$

The Taylor series, which has zero radius of convergence, is

$$f(t) = 1 - t^2 + t^4 - \tfrac{17}{15}t^6 + \tfrac{31}{21}t^8 - \tfrac{691}{315}t^{10} + \ldots, \tag{3.5}$$

and $f(t)$ is a monotonically decreasing function of t^2 which goes like t^{-4} at large t. This $f(t)$ may be interpreted as the Lagrangian velocity correlation $\langle v(0)\,v(t)\rangle$

of a particle which starts at $t = 0$, $x = 0$ in a prescribed, static Eulerian velocity field in one dimension, $u(x) = A\sin(\pi x) + B\cos(\pi x)$, where A and B are independent Gaussian variables, $\langle A^2 \rangle = \langle B^2 \rangle = 0{\cdot}5$, and $\langle\ \rangle$ denotes ensemble average (Orszag, private communication).

Here $\rho(a)$ goes like $\exp(-|a|^{\frac{2}{3}})$ at large a, and the counter-example given in §2 shows, therefore, that $f(t)$ is not uniquely determined by the coefficients c_{2n} even if we use the knowledge that $\rho(a)$ is positive everywhere. There is no $w(a)$ such that the orthonormal functions are complete and, at the same time,

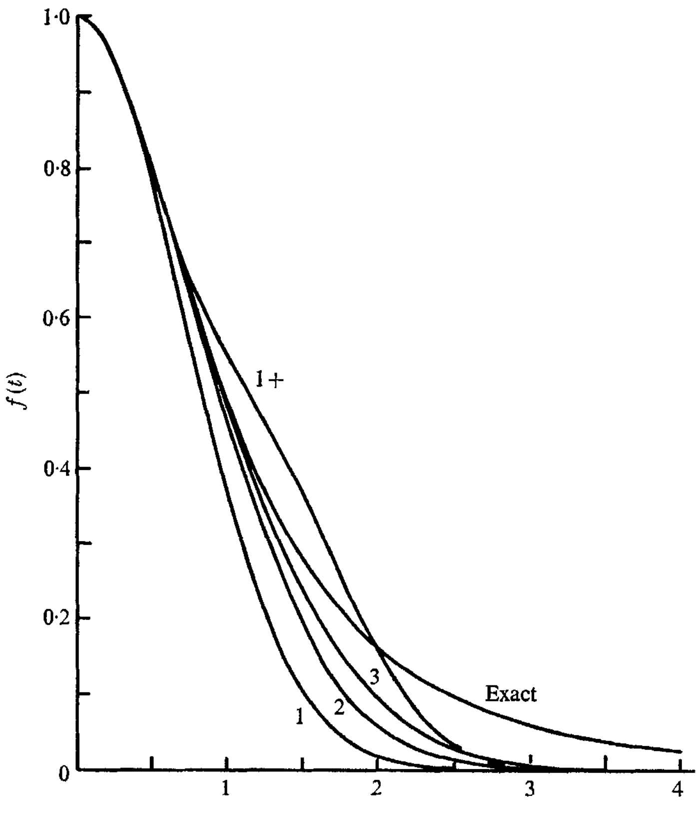

FIGURE 5. Approximants $f_r(t)$ to the $f(t)$ of (3.4), (3.5), with weight taken as the best fitting sum of Gaussians (3.2). Curves 1, 1+, 2, 3 show $r = 1, 2, 3, 5$.

(2.8) converges. However, if we repeat the procedure used above, taking an adjustable $w(a)$ of form (3.2) and minimizing the projection on the higher orthogonal functions, the results are puzzlingly good. Again (3.3) gives an exact match to the Taylor series through order $2r$ (at least for odd $r \leqslant 5$). The values found for A_i, β_i and $\rho_r(0)/\rho(0)$ are now as follows:

r	$\rho_r(0)/\rho(0)$	A_1	β_1	A_2	β_2	A_3	β_3
1	0·707	1	1	—	—	—	—
3	0·810	0·907	0·824	0·093	2·03	—	—
5	0·854	0·812	0·740	0·186	1·69	0·0023	3·16

Figure 5 compares $f(t)$ with $f_r(t)$ for $r = 1, 2, 3, 5$. The approximant $f_2(t)$ is obtained using the single-Gaussian $w(a)$ from $r = 1$. It already shows symptoms of the divergent behaviour of the fixed-weight expansion. On the other hand, the curves for $r = 1, 3, 5$ appear at least consistent with convergence to the correct $f(t)$.

Since $f(t)$ goes as t^{-4} at large t, while any expansion based on Gaussians can only give Gaussian fall-off at large enough t, we may hope to improve the results just given by some more appropriate choice. For this purpose, let us replace the underlying function $(4\pi)^{-\frac{1}{2}}\beta_i^{-1}\exp(-a^2/4\beta_i^2)$ in (3.2) by

$$\tfrac{1}{8}\beta^{-1}(1+|a|/2\beta_i)\exp(-|a|/2\beta_i),$$

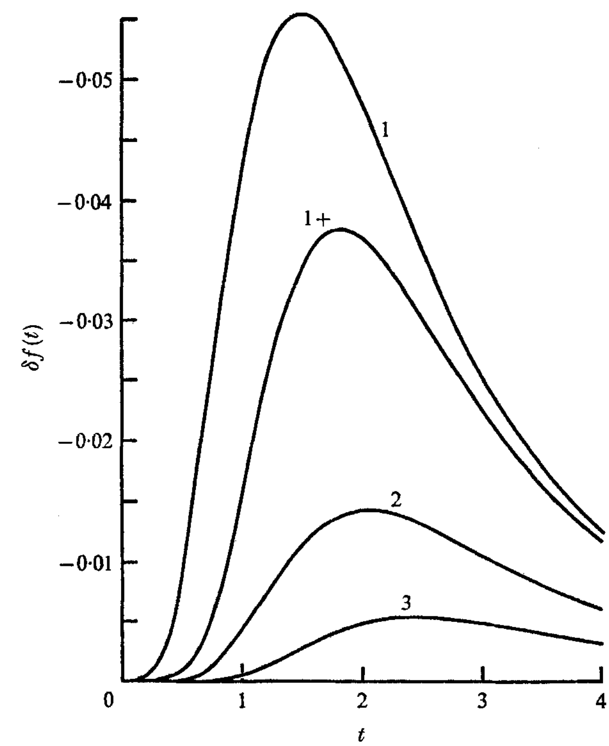

FIGURE 6. Errors $\delta f_r(t)$ for $f(t)$ of (3.4), (3.5) and weight taken as best-fitting transform of (3.6). Curves 1, 1+, 2, 3 show $r = 1, 2, 3, 5$.

which has the cosine transform $(1+\frac{1}{2}\beta_i^2t^2)^{-2}$. As it happens, this choice, which is about the simplest with the correct qualitative behaviour at large t, is related to Gaussian weight by an integration over width parameter that precisely corresponds to the p integration in (3.4). Thus we find again that the Taylor series is matched exactly through order $2r$, now by

$$f_r(t) = \sum_{i=1}^{\frac{1}{2}(r+1)} A_i(1+\tfrac{1}{2}\beta_i^2t^2)^{-2}. \tag{3.6}$$

The numerical results for $\rho_r(0)/\rho(0)$, A_i and β_i are precisely those given after (3.3). Figure 6 shows the errors $\delta f_r(t) - f(t)$ for $r = 1, 2, 3, 5$. There is a marked improvement over the Gaussian fit. Also, the $r = 2$ curve gives a smooth improvement over $r = 1$, suggesting that, even though the expansion on fixed $w(a)$ is still divergent, it has the nature of an asymptotic expansion whose optimum cut-off lies beyond the lowest term.

The results described above suggest the possibility that, implicitly, the adjustable-weight procedure uses additional information which did not enter the convergence theory of §2. Perhaps the apparent convergence to (3.4), (3.5) with Gaussian weight is associated with the existence of the integral representation (3.4), wherein $f(t)$ appears as a weighting of functions with finite radius of convergence in t. We may note that if the p integral is cut off at any finite value, or is approximated in some more elaborate fashion by a weighting with finite cut-off (corresponding to cut-off distributions of the amplitudes A and B in the underlying particle-motion), the sufficient conditions for the convergence theory of §2 are satisfied. It may be that the approximation method we have just explored effectively does this. We shall return to this question in §4, in the context of a particle-diffusion problem more relevant to actual turbulent flow.

4. A three-dimensional diffusion problem

Consider a three-dimensional, incompressible, homogeneous, isotropic, and statistically stationary ensemble of velocity fields $\mathbf{u}(\mathbf{x}, t)$. The position $\mathbf{y}(t)$ and velocity $\mathbf{v}(t)$ of a particle, released at $\mathbf{y}(0) = \mathbf{x}$, which moves with the fluid are determined by

$$\mathbf{v}(t) = \mathbf{u}(\mathbf{y}, t), \quad \mathbf{y}(t) = \mathbf{x} + \int_0^t \mathbf{v}(s)\, ds. \tag{4.1}$$

Suppose that we wish to evaluate the Lagrangian velocity correlation

$$U_L(t) = \tfrac{1}{3}\langle \mathbf{v}(0) \cdot \mathbf{v}(t) \rangle / v_0^2, \tag{4.2}$$

where $\langle\ \rangle$ denotes ensemble average and v_0 is the root-mean-square value of the component of $\mathbf{u}(\mathbf{x}, t)$ along any axis. A formal series expansion of $U_L(t)$ in powers of t may be obtained straightforwardly as follows. Expand $\mathbf{v}(t)$ as a Taylor series with unknown coefficients and expand $\mathbf{u}(\mathbf{y}, t)$ as a four-dimensional Taylor series about $\mathbf{y} = \mathbf{x}$, $t = 0$. The coefficients of the latter series are the derivatives of the Eulerian field $\mathbf{u}(\mathbf{x}, t)$. The coefficients in the expansion of $\mathbf{v}(t)$ can then be expressed as products of derivatives of $\mathbf{u}(\mathbf{x}, t)$, evaluated at $(\mathbf{x}, 0)$, by substituting both series into (4.1) and requiring that like powers of t balance on the two sides of each equation. If the ensemble of realizations $\mathbf{u}(\mathbf{x}, t)$ is now specified by giving all moments of $\mathbf{u}$, the Taylor series for $U_L(t)$ results from substituting the series for $\mathbf{v}(t)$ into (4.2). The coefficients of successive powers of t in the U_L expansion are successively higher-order moments of the multivariate distribution of $\mathbf{u}(\mathbf{x}, t)$ and its derivatives. Only even-order moments, and even powers of t, survive in the final result, because of the statistical symmetry conditions we have imposed.

The evaluation is easiest when the $\mathbf{u}(\mathbf{x}, t)$ distribution is prescribed to be multivariate-normal, so that all moments can be expressed as functions of the covariance. Let us take the concrete case where

$$\langle \mathbf{u}(\mathbf{x}+\mathbf{r},t)\cdot\mathbf{u}(\mathbf{x},t')\rangle = \exp\left[-\tfrac{1}{2}\omega_0^2(t-t')^2\right]\int_0^\infty E(k)(\sin(kr)/kr)\,dk,$$

$$E(k) = 16(2/\pi)^{\frac{1}{2}}v_0^2k_0^{-5}k^4\exp(-2k^2/k_0^2). \tag{4.3}$$

Here k_0 and ω_0 are the characteristic wave-number and frequency scales of the Eulerian field and $E(k)$ is the energy spectrum, which peaks at $k = k_0$ and is normalized by $\int_0^\infty E(k)\,dk = 3v_0^2/2$. The resulting series for $U_L(t)$ is

$$U_L(t) = 1-(\omega_0^2+\tfrac{5}{4}v_0^2k_0^2)(t^2/2!)+(3\omega_0^4+\tfrac{35}{4}\omega_0^2v_0^2k_0^2+\tfrac{235}{32}v_0^4k_0^4)(t^4/4!)-\dots. \tag{4.4}$$

Corrsin (1952), extending earlier work of Taylor (1921), showed that the effective eddy diffusivity exerted at time t on a very low wave-number cloud of marked particles introduced at time $t = 0$ is $v_0^2\int_0^t U_L(s)\,ds$. We may extend the concept of effective eddy diffusivity to general-wave-number marked-particle distributions as follows. The density of marked particles in any realization obeys

$$(\partial/\partial t)\,\psi(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) = 0. \tag{4.5}$$

Let $\psi(\mathbf{x},0) = \overline{\psi}_k\cos(\mathbf{k}\cdot\mathbf{x})$ in every realization. It follows from the statistical symmetry conditions that the mean field $\overline{\psi}(x,t)$ at later times has the form $\overline{\psi}_k(t)\cos(\mathbf{k}\cdot\mathbf{x})$. Let us define $\gamma_k(t)$ by

$$(d/dt)\,\overline{\psi}_k(t)+v_0^2k^2\int_0^t\gamma_k(t-s)\,\overline{\psi}_k(s)\,ds = 0. \tag{4.6}$$

To see the significance of (4.6), consider its Laplace transform,

$$\chi_k(\alpha) = \overline{\psi}_k/[\alpha+v_0^2k^2\Gamma_k(\alpha)], \tag{4.7}$$

where $\chi_k(\alpha)$ and $\Gamma_k(\alpha)$ are the transforms of $\overline{\psi}_k(t)$ and $\gamma_k(t)$, respectively. If (4.5) were replaced by a simple molecular diffusion equation, we would have

$$\chi_k(\alpha) = \overline{\psi}_k/[\alpha+\eta k^2], \tag{4.8}$$

where η is the molecular diffusivity, instead of (4.7). Thus $v_0^2\Gamma_k(\alpha)$ plays the role of a frequency-dependent effective diffusivity. In particular, returning to (4.7), we have

$$\chi_k(0) = \int_0^\infty\overline{\psi}_k(t)\,dt = \overline{\psi}_k/(k^2\kappa_k), \tag{4.9}$$

where

$$\kappa_k = v_0^2\Gamma_k(0) = v_0^2\int_0^\infty\gamma_k(t)\,dt \tag{4.10}$$

is the effective long-time eddy diffusivity at wave-number k.

As $k\to 0$, the rate of change of $\overline{\psi}_k(s)$ approaches zero, so that (4.6), together with Corrsin's results, implies that $\gamma_0(t) = U_L(t)$.

We may express $\gamma_k(t)$ as a formal Taylor series in t by expanding $\psi(\mathbf{x},t)$ as a Taylor series in t in each realization, averaging to get the Taylor series for $\overline{\psi}_k(t)$, and then substituting the latter into (4.6) in order to determine the coefficients in the series for $\gamma_k(t)$. The analysis is facilitated by starting with the spatial Fourier transform of (4.5). The final result for the multivariate-normal distribution with covariance (4.3) is

$$\gamma_k(t) = 1-(\omega_0^2+\tfrac{5}{4}v_0^2k_0^2+2v_0^2k^2)(t^2/2!)+(3\omega_0^4+\tfrac{35}{4}\omega_0^2v_0^2k_0^2 + 14\omega_0^2v_0^2k^2+\tfrac{235}{32}v_0^4k_0^4+\tfrac{95}{4}v_0^4k_0^2k^2+10vk_0^4k^4)(t^4/4!)-\ldots \quad (4.11)$$

For $k = 0$, (4.11) reduces to (4.4), as it should.

Only the numerical coefficients in (4.4) and (4.11) are consequences of the multivariate-normal distribution of $\mathbf{u}(\mathbf{x}, t)$. Any other $\mathbf{u}$ distribution which satisfies the statistical symmetries and has moments which can be expressed as series in frequency and wave-number parameters ω_0^2 and k_0^2, yields the same functional forms, for each power of t^2, as in (4.4) and (4.11).

So far, (4.4) and (4.11) are only formal series representations. To elucidate the analyticity properties of $U_L(t)$ and $\gamma_k(t)$, consider first a realization in which $\mathbf{u}(\mathbf{x}, t)$ is an analytic function of the three space co-ordinates, and of time, over all space-time. Any $\mathbf{u}(\mathbf{x}, t)$ whose 4-dimensional Fourier transform does not extend to infinite frequency or wave-number satisfies this criterion; in particular, it is satisfied if $\mathbf{u}(\mathbf{x}, t)$ consists of a finite number of discrete 4-dimensional Fourier components. For such a realization, we may conclude immediately that $\mathbf{v}(t)$ is analytic about $t = 0$, with at least a finite radius of convergence of its Taylor series, and that the same is true for $\psi(\mathbf{x}, t)$, provided that $\psi(\mathbf{x}, 0)$ is analytic in the space variables over all space. This follows from the fact that (4.1) and (4.5) are analytic equations, involving, now, only analytic functions of analytic functions.

However, we may argue that the radius of convergence of the Taylor series in t for either $\mathbf{v}(t)$ or $\psi(\mathbf{x}, t)$ is typically finite rather than infinite in an analytic realization of $\mathbf{u}(\mathbf{x}, t)$. Suppose that $\mathbf{u}(\mathbf{x}, t)$ consists of a finite number of Fourier components and consider the behaviour of (4.1) for complex t. We get complex $\mathbf{y}$, which means complex arguments in the sine and cosine functions that give $\mathbf{u}$ along the trajectory. This gives the possibility of running away to complex infinity along some line in which $\mathbf{u}$ increases exponentially. The behaviour in this case would resemble the solution of the equation $d|y|/d|t| = \exp|y|$, whose solutions blow up in finite time. We therefore expect that, at least for some choices of $\mathbf{x}$, $\mathbf{v}(t)$ will display singularities at finite, complex t, implying finite radius of convergence of the Taylor series. This behaviour can be directly verified for the one-dimensional velocity field which led to (3.4) and (3.5), and there seems no reason why increased complexity of the fields, incompressibility, or higher dimensionality should change this behaviour. Similar arguments (together with the physical interpretation of $\psi(\mathbf{x}, t)$ as a density of particles) suggest that the radius of convergence of the Taylor series in t for $\psi(\mathbf{x}, t)$ is typically only finite also. We are not attempting to prove this, only to point out, rather, that infinite radius of convergence cannot be assumed.

Now consider a stationary, isotropic, homogeneous ensemble of velocity fields in which $\mathbf{u}(\mathbf{x}, t)$ for each realization consists of a finite number of Fourier components and such that, in each realization, the amplitude of any Fourier component, the number of components, the maximum wave-number, and the maximum frequency all have upper bounds which are the same for all realizations. Then it follows from analyticity of the realizations that $U_L(t)$ and $\overline{\psi}_k(t)$ are analytic about $t = 0$ so that their Taylor series have at least finite radius of convergence. Further, it follows from the defining equation (4.6) that $\gamma_k(t)$ then must also be analytic in at least a finite region containing $t = 0$. Now it follows from statistical stationarity of $\mathbf{u}$ and incompressibility that $U_L(t)$ is the covariance of a real stationary process (Lumley 1962) and has a positive Fourier transform. Also, $\gamma_k(t)$ has a positive transform. This follows from (4.7) and the fact that $\overline{\psi}_k(t)/\overline{\psi}_k$ is the mean response function for wave-vector k in a conservative, linear system and must itself have a positive Fourier transform. Finally, if the ensemble smears over a band of frequencies of the Eulerian field, and the latter has a finite decorrelation time, then it is clear physically that $U_L(t)$ and $\gamma_k(t)$ have smeared spectra which go smoothly to zero at infinite frequency. We conclude that, for such a bounded $\mathbf{u}$ distribution, the series corresponding to (4.4) and (4.11) satisfy the conditions of statement (2) of §2, for uniform convergence of (2.7).

But if the radius of convergence for $\mathbf{v}(t)$ is finite in some finite-measure set of realizations, then we must expect $U_L(t)$ and $\gamma_k(t)$ to be non-analytic at $t = 0$ for a normal distribution or for any distribution in which amplitudes are not bounded. This is because the radius of convergence in a realization should typically have some sort of inverse relation to velocity amplitude. To handle an unbounded distribution, we could represent it as the limit of a sequence of bounded distributions and use the orthogonal expansion method to converge on $U_L(t)$ and $\gamma_k(t)$ for successive members of the sequence, until the converged answers differ little enough to satisfy us. This, in fact, follows the physics, since unbounded distributions do not occur in nature. However, in view of the success we had with (3.5), it is interesting to see what the same procedure used there gives when applied directly to (4.4) and (4.11) for the normal distribution.

Figures 7–10 show approximants $r = 1$ and $r = 2$ for $U_L(t)$ and $\kappa_k = \int_0^\infty \gamma_k(t)\,dt$ obtained from (4.4) and (4.11) by using, in each case, that Gaussian weight which makes $b_2 = 0$. Results are shown for $\omega_0 = v_0 k_0$ (characteristic Eulerian frequency = characteristic eddy-circulation frequency) and for $\omega_0 = 0$ (frozen Eulerian field). In each case, the results are compared with computer simulations of the particle diffusion (Kraichnan 1970*b*). The statistical uncertainty in the simulations is shown in the plots.

Again, it is puzzling that the results are so good. The case $\omega_0 = v_0 k_0$ is perhaps not so mysterious. If $\omega_0 \gg v_0 k_0$, then $U_L(t) \approx \exp(-\frac{1}{2}\omega_0^2 t^2)$; that is, the particles do not have time to move appreciably in a characteristic time of the Eulerian field, and the Eulerian and Lagrangian velocity correlations are indistinguishable. It is also not hard to see that $\gamma_k(t) \approx \exp(-\frac{1}{2}\omega_0^2 t^2)$, in this limit. Thus it is reasonable that, if $\omega_0 \geqslant v_0 k_0$, $U_L(t)$ and $\gamma_k(t)$ are not far from Gaussian in form, and even an orthogonal expansion based on a fixed Gaussian weight, though surely divergent eventually, may give good approximations in the sense of an asymptotic expansion. But this does not explain why we get good approximations for $\omega_0 = 0$.

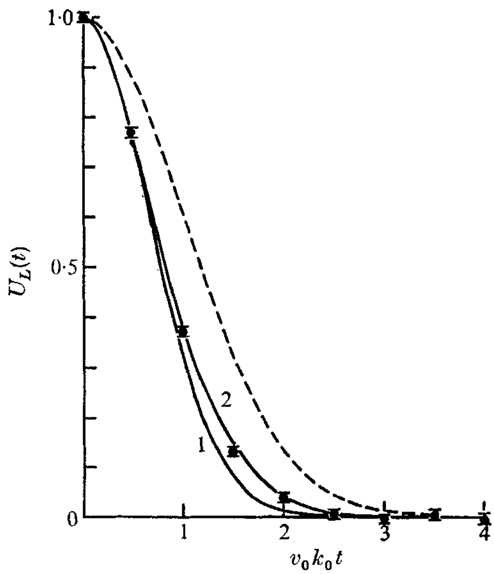

FIGURE 7

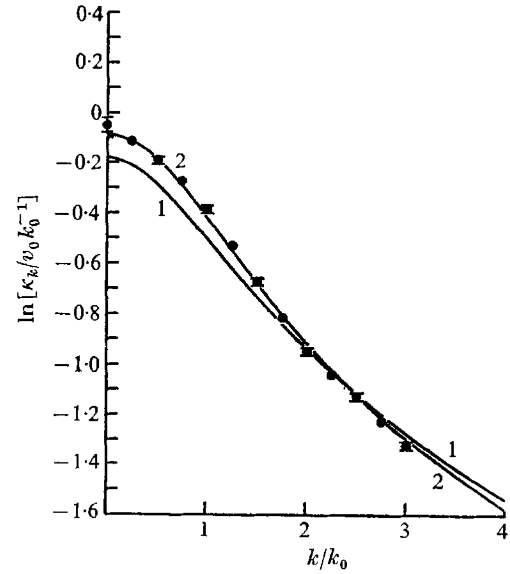

FIGURE 8

FIGURE 7. Approximants $r = 1$ (curve 1) and $r = 2$ (curve 2) to Lagrangian velocity correlation $U_L(t)$ with $\omega_0 = v_0 k_0$. Points are computer simulation results and bars give probable errors. Dashed curve is the Eulerian velocity correlation $\exp(-\frac{1}{2}\omega_0^2 t^2)$.

FIGURE 8. Approximants $r = 1$ and $r = 2$ to k-dependent eddy diffusivity κ_k for $\omega_0 = v_0 k_0$. Points are computer simulation results.

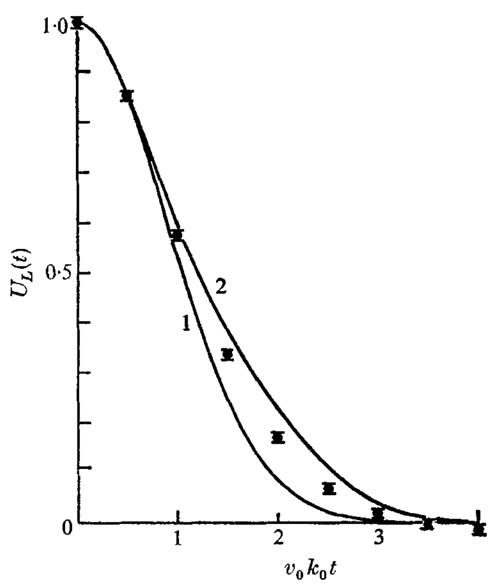

FIGURE 9

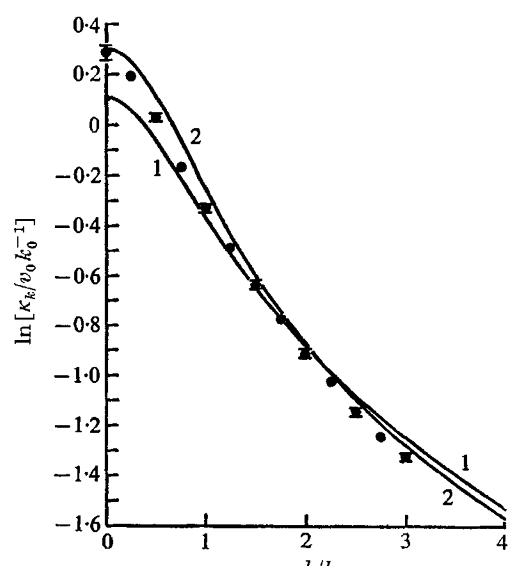

FIGURE 10

FIGURE 9. Approximants $r = 1$ and $r = 2$ for $U_L(t)$ with $\omega_0 = 0$ (frozen Eulerian velocity field) compared with computer simulation results.

FIGURE 10. Approximants $r = 1$ and $r = 2$ for κ_k with $\omega_0 = 0$, compared with computer simulation results.

An observation relevant both here and in the case of (3.5) is that, to the orders we have evaluated, the expansions use only relatively low-order moments of the underlying velocity fields. It is very likely that there are bounded distributions, for which the convergence theory (with proper weight) is solid, that yield Taylor series identical with (3.5), (4.4) and (4.11) through the orders explicitly examined. To say this somewhat differently, there may be a genuinely irreducible ambiguity associated with the non-analyticity of the expansions, but the non-analytic parts of $U_L(t)$ and $\gamma_k(t)$, which we cannot determine uniquely from the expansions, may be effectively small, in the sense that both the values of $U_L(t)$ and $\gamma_k(t)$, and the low-order terms of their Taylor expansions, are insensitive to the form of the tails of the underlying velocity distributions.

The question arises, at this point, of what, particularly, the orthogonal expansion method has to do with statistics and turbulence, in view of the fact that the conditions for convergence of the expansion for averages rest upon similar conditions for unaveraged quantities. The answer lies in the expectation that averages are smooth, simple-looking quantities, while the unaveraged functions are complicated. The computer simulations for the present problem show that $\mathbf{v}(t)$, in a typical realization, oscillates perpetually and irregularly with an amplitude $\sim v_0$. No simple choice of $w(a)$ would be simply related to its spectrum, and convergence of the orthogonal expansion, even when assured, would likely be slow and uneven (cf. the discussion of the expansion for $f(t) = \cos t$ in §3). On the other hand, we see from figures 7 and 9 that $U_L(t)$ is smooth and uncomplicated.

5. Damped systems and coupling-strength expansions

The diffusion problem treated in §4 is special in two ways. First, the equations of motion are undamped and reversible in time. Second, $U_L(t)$ and $\gamma_k(t)$ have everywhere-positive spectra. Direct application of the orthogonal expansion method to Taylor series in time runs into difficulties in damped systems. Consider $f(t) = e^{-t}$ $(t \geqslant 0)$. In order to take the Fourier transform, we must define $f(t)$ for all t, and this means a discontinuity of some order at some $t \leqslant 0$ (implying a transform that goes like a power at large $|a|$) or an essential singularity at some $t \leqslant 0$. We can circumvent this trouble by working with a transformed variable, for example $s^2 = t$ or $s^2/(1+s^2)^{\frac{1}{2}} = t$, but this seems unnatural unless the damping contributes strongly to the dynamics.

On the other hand, the orthogonal expansion method should work on the t series for many problems where the dynamics are reversible. Any statistically stationary process obeying analytic equations can be handled like the diffusion example, if the transform $\rho(a)$ of the desired statistical function is bounded at large $|a|$. More generally, when the underlying process is conservative, but not statistically stationary, it may still be possible, starting with a bounded statistical distribution and analytic equations of motion, to bound $f(t)$, demonstrate analyticity in a strip about the real t-axis, and obtain bounds for all t of the form $|d^nf(t)/dt^n| < n!\,q^{-n}C$, with some positive q and C. If the statistical distribution smears in some reasonable way over underlying frequency parameters of the system, $\rho(a)$ should have reasonable smoothness properties at large $|a|$. Then statement (1′) of §2 should be satisfied, provided $f(t)$ is so constructed that $\int_{-\infty}^{\infty}[f(t)]^2dt$ exists.

An example where use of the Taylor series in time seems appropriate is the calculation of the constant of proportionality between scalar spectrum level and rate of transfer of scalar variance in Batchelor's k^{-1} law spectral range for convection of a passive scalar (Batchelor 1959). Here we set up again a prescribed isotropic, homogeneous, stationary, incompressible ensemble of velocity fields, formed as in §4, from realizations in which amplitudes, wave-numbers, and frequencies are bounded. At $t = 0$, we assume that there exists a k^{-1} spectrum of scalar at wave-numbers $k \gg k_0$, where k_0 characterizes the velocity field, and that there is no initial correlation between velocity and scalar fields. A Taylor series is easily developed for the rate at which scalar variance is transported from below some given k in the k^{-1} range to above. This series contains only odd powers of t. If Batchelor's ideas are correct, the characteristic time or build-up of this transport is the reciprocal vorticity of the velocity field, and is independent of k. Then the transport at any k in the range should rise to an equilibrium value while the spectrum level itself is unchanged at all t. The time derivative of the rate of transport is then the candidate for the orthogonal-expansion treatment, with the asymptotic $t = \infty$ transport rate determined, as in previous examples, as $\int_0^{\infty} f(t)\,dt$.

With suitable modifications and precautions, a similar technique could be used to compute Kolmogorov's inertial-range constant, provided, again, that the assumed qualitative physical ideas are correct.

The orthogonal expansion method can be applied in a general way to damped systems by expanding not in t but in a strength parameter placed in front of the terms that are non-linear in stochastic quantities. In turbulence applications, this means a Reynolds or Péclet number expansion. To illustrate the nature of the strength-parameter expansion, consider the two classes of systems,

$$\dot{y}_i + \nu_i y_i = \lambda \Sigma_j A_{ij} y_j \tag{5.1}$$

and

$$\dot{y}_i + \nu_i y_i = \lambda \Sigma_{jk} A_{ijk} y_j y_k, \tag{5.2}$$

where the y's are dynamic variables, the ν_i are positive or zero damping constants, the A's are (possibly stochastic) coefficients and λ, the strength parameter, takes the value one in the case of eventual interest. Turbulent convection (with stochastic A's representing a frozen velocity field) is an example of (5.1), while Navier–Stokes turbulence (with non-stochastic A's) is an example of (5.2).

If all $\nu_i = 0$, the Taylor series in t for y_i is actually a series in λt, and the results of the orthogonal expansion method are indifferent to whether λ or t is taken as expansion variable. Suppose next that $\nu_i = \nu$, the same for all i. Then (5.1) yields $y_i(t) = e^{-\nu t} g_i(\lambda t)$, where $g_i(\lambda t)$ is the λt expansion of $y_i(t)$ for the case $\nu = 0$. The result of applying the orthogonal expansion method to the λ series is now the same as if we applied it to the $\nu = 0$ problem, using the same weight, and then, at the end, multiplied each convergent by $e^{-\nu t}$. Note that, just as λ would appear as a parameter in the weight $w(a)$ if we used t as expansion parameter, so now we will have t as a parameter in $w(a)$ when we use the λ expansion (a now being the Fourier-transform pair of λ). If $\nu_i = \nu$ for all i, then (5.2) yields $y_i(t) = e^{-\nu t} g_i(\lambda t_*)$, where $g_i(\lambda t)$ is the t-expansion of $y_i(t)$ for $\nu = 0$ and $t_* = (1-e^{-\nu t})/\nu$. Thus the result of applying the orthogonal expansion method to the λ expansion in this case is the same as if we applied it in the case $\nu = 0$ at time t_* and then multiplied the convergents by $e^{-\nu t}$.

In all the cases just discussed, it is clear that the orthogonal expansion method applied to the λ expansion will succeed if it succeeds for the t expansion of the undamped system. However, the behaviour is less clear for the non-trivial cases where the ν_i are not all the same. In general, success of the expansion requires that the function of interest have a sufficiently healthy dependence on λ over the whole real λ-axis, in the sense we have discussed previously for the t-expansion. Particularly in the Navier–Stokes turbulence case, this is a rather uncomfortable requirement, even if satisfied, because it makes the nature of convergence for finite Reynolds number depend on behaviour out to infinite Reynolds number. To avoid this we can introduce a new strength parameter μ, such that $-\infty \leqslant \mu \leqslant \infty$ maps onto $-1 \leqslant \lambda \leqslant 1$. There are many ways to do this. Two simple possibilities are

$$\lambda = \mu/(1+\mu^2)^{\frac{1}{2}}, \tag{5.3}$$

$$\lambda = \sin(\tfrac{1}{2}\pi\mu). \tag{5.4}$$

If (5.3) is used, the power series in μ can be obtained by expanding the terms of the power series in λ (obtained from the equations of motion), and we note that the radius of convergence of the μ series is non-zero if that of the λ series is. With (5.4), all functions are periodic in μ, and all the a integrals in §2 must be replaced by discrete sums. (This applies, in particular, to (2.5).) In either case we now need only examine the dynamical behaviour for $0 < |\lambda| < 1$. Suppose we wish to calculate the energy of a decaying isotropic turbulence at some time t. We expand it in λ, convert to a power series $E(\mu)$, using (5.3), apply the orthogonal expansion method to $f(\mu) = dE(\mu)/d\mu$, and, finally, find the approximants to the energy as the approximants to $\int_0^\infty f(\mu)\,d\mu$.

6. Expansion about the direct-interaction approximation

It is not difficult to pose turbulence problems in which convergence of the orthogonal expansion method is unacceptably slow if it is applied to the power series in time or in strength parameter. Consider, for example, the decay of isotropic turbulence when the initial distribution has all odd-order moments zero and the initial energy is all confined below some wave-number k_0. The Taylor series in time (or strength parameter) for the energy in a wave-number $k > 2^n k_0$ has all zero coefficients until order $2n+2$, while the labour of computing coefficients goes up rapidly with order. Clearly it is desirable, in such a problem, to expand about an approximation which incorporates the many-step energy cascade at the start.

The direct-interaction approximation is a logical basis for such an expansion because it represents exactly a model dynamical system and has given good numerical agreement with isotropic decay experiments (Orszag 1970*a*). The model system (Kraichnan 1961) is obtained by considering an infinite collection of similar flow systems, setting up collective co-ordinates over the collection (labelled by $\alpha, \beta, \gamma, \ldots$), and then altering, in a random fashion, the dynamical couplings of the collective co-ordinates given by the Navier–Stokes equation. The difference between the original Navier–Stokes system and the final model system is expressed by a set of coupling factors $\phi_{\alpha\beta\gamma}$, which all have the value $+1$ in the Navier–Stokes system and take the values ± 1 at random in the model.

A model system which bridges between the direct-interaction approximation and the exact dynamics can now be constructed by taking

$$\phi_{\alpha\beta\gamma} = \frac{\pm 1}{(1+\lambda^2)^{\frac{1}{2}}} + \frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}}, \tag{6.1}$$

where, again, the plus and minus signs are taken at random. Then $\lambda = 0$ gives the direct-interaction approximation and $\lambda = \infty$ is the exact dynamics. With this model, it is not hard to see that (in the language of Kraichnan 1961) all irreducible diagrams of order $2n$ ($n \geqslant 2$) in the expansions of the equations for Green's function and covariance appear with a factor $[\lambda^2/(1+\lambda^2)]^n$, while the second-order diagram, which is the same for both direct-interaction model and exact dynamics, is unchanged.

Given an initial ensemble, the model equations specified by (6.1) can be solved by iteration methods to yield an explicit expansion in powers of λ for any statistical function at later times. We could then hope to apply the orthogonal expansion method on the basis that, since we are always dealing with an actual dynamical system, there should be a non-singular variation of statistical functions with λ over the whole λ range. However, there is a serious difficulty in principle at the present state of the convergence theory. For $\lambda > 0$, the actual Navier–Stokes non-linear terms appear in the dynamical equations with finite strength and, as a result, we must expect that the radius of convergence of the power series in λ cannot be > 0 if that of the strength-parameter (Reynolds number) expansion is zero. But, extrapolating from the arguments of §4 (cf. Kraichnan 1966), we must assume that the latter radius is zero for any unbounded distribution. The trouble now is that the bridging model (6.1) makes bounded distributions impossible. An infinite collection is coupled, and, no matter what distribution is taken initially for the individual systems, the random coupling factors give a Gaussianly distributed input to each system from interaction with all the others.

We therefore propose working with an alternative expansion about the direct-interaction approximation, based on a different model equation, which avoids this difficulty and is, in addition, much more transparent. This expansion is closely related to one proposed by Phythian (1969). Consider the general non-linear system

$$\dot{y}_i + \nu_i y_i = \Sigma_{jk} A_{ijk} y_j y_k, \quad A_{ijk} = A_{ikj}, \tag{6.2}$$

which is (5.2), written now without strength parameter. Assume, for simplicity, that the initial statistics and the structure of the A's is such that $\langle y_i(t)\, y_j(t')\rangle = 0$

$(i \neq j)$ for all t and t' (e.g. when the y's are Fourier modes of isotropic turbulence). Then the direct-interaction equations are (Kraichnan 1958)

$$(\partial/\partial t + \nu_i) G_i(t,t') = 4\Sigma_{jk} A_{ijk} A_{jki} \int_{t'}^{t} G_j(t,s) Y_k(t,s) G_i(s,t')\,ds, \quad G_i(t',t') = 1, \tag{6.3}$$

$$(\partial/\partial t + \nu_i) Y_i(t,t') = 4\Sigma_{jk} A_{ijk} A_{jki} \int_0^{t} G_j(t,s) Y_k(t,s) Y_i(s,t')\,ds + 2\Sigma_{jk}(A_{ijk})^2 \int_0^{t'} G_i(t',s) Y_j(t,s) Y_k(t,s)\,ds. \tag{6.4}$$

Here $Y_i(t,t') = \langle y_i(t)\, y_i(t')\rangle$ and $G_i(t,t')$ is the mean response function for infinitesimal perturbations of y_i. That is, $G_i(t,t') = \langle \delta y_i(t)/\delta f_i(t')\rangle$, where f_i is an infinitesimal forcing term added to the right-hand side of (6.2). Equations (6.3) and (6.4) can be integrated forward from $t = t' = 0$ if the $Y_i(0,0)$ are prescribed. Additional terms must be added to (6.3) and (6.4) if the initial y distribution has non-zero means $\langle y_i\rangle$.

The alternative model equation is

$$\dot{y}_i(t) + \nu_i y_i(t) + (1-\lambda^2) \int_0^{t} \eta_i(t,s)\, y_i(s)\,ds = (1-\lambda^2)^{\frac{1}{2}} q_i(t) + \lambda \Sigma_{jk} A_{ijk} y_j(t) y_k(t), \tag{6.5}$$

where
$$\eta_i(t,s) = -4\Sigma_{jk} A_{ijk} A_{jki} G_j(t,s) Y_k(t,s), \tag{6.6}$$

and $q_i(t)$ is a stochastic force, statistically independent of the initial y distribution, whose covariance is

$$\langle q_i(t)\, q_i(t')\rangle = 2\Sigma_{jk}(A_{ijk})^2 Y_j(t,t') Y_k(t,t'). \tag{6.7}$$

One way to realize the q's is

$$q_i(t) = \Sigma_{jk} A_{ijk}\, \xi_j(t)\, \xi'_k(t), \tag{6.8}$$

where ξ and ξ' are stochastic variables, statistically independent of each other and of the initial y distribution, and such that

$$\langle \xi_i(t)\, \xi_j(t')\rangle = \langle \xi'_i(t)\, \xi'_j(t')\rangle = \delta_{ij} Y_j(t,t'). \tag{6.9}$$

The appearances of averages in (6.5) means that it also may be thought of as representing a dynamical coupling of an infinite collection of systems, namely, all the realizations in the ensemble. However, in contrast to the original direct-interaction model, the couplings are non-random. For $\lambda = 1$, (6.5) reduces to (6.2). For $\lambda = 0$, it is a dynamically linear equation, since change of any values in a single realization has infinitesimal effect on averages over the infinite ensemble. The direct-interaction response function equation (6.3) follows immediately from (6.5), (6.6) at $\lambda = 0$, in view of this. From the dynamical linearity we have also

$$\langle y_i(t')\, q_i(t)\rangle = (1-\lambda^2)^{\frac{1}{2}} \int_0^{t'} G_i(t',s) \langle q_i(s)\, q_i(t)\rangle\,ds \quad (\lambda = 0), \tag{6.10}$$

whence, multiplying (6.5) by $y_i(t')$ and averaging, using (6.7), we obtain (6.4). Thus we recover the complete direct-interaction equations, for $\lambda = 0$, without restrictions, other than those already stated, on the q-distribution; it need not be Gaussian.

If the coefficients satisfy
$$A_{ijk} + A_{jki} + A_{kij} = 0, \tag{6.11}$$

(6.2) conserves energy $\Sigma_i(y_i)^2$ if the ν_i vanish. This property is preserved in average by the direct-interaction equations and, consequently, the model equation (6.5) conserves energy in the mean at $\lambda = 0$. That is, the total energy of the ensemble is conserved. If we take the q_i to be Gaussian variables, then (6.10) holds at non-zero λ, despite the dynamical non-linearity (cf. Novikov 1964). In this case, the model equation conserves energy in the mean for the whole range $0 \leqslant \lambda \leqslant 1$. The form of (6.5) recalls the interpretation of the direct-interaction approximation in terms of a dynamical damping and an input to a given mode from all others (Kadomtsev 1965) and also is close in spirit to the approaches of Edwards (1964) and Herring (1966). By (6.11) and (6.6), the quantity $\eta_i(t,s)$, a dynamical damping with memory, is typically (but not always) positive. The realizability of the direct-interaction approximation for $Y_i(t,t')$ follows immediately from the representation (6.5) of the approximation by an amplitude equation.

If the y_i and q_i are written as power series in λ, then (6.5) can be solved by iteration to yield, on averaging, an explicit power series in λ for any statistical function of interest, and with any choice of initial y-distribution. We may then transform according to, say, (5.3) or (5.4), and apply the orthogonal expansion method to the power series in μ. We expect non-singular, bounded behaviour over the whole μ range, and analyticity in a strip about the real axis, if the q- and initial y-distributions are bounded. But there is still a trouble, with the q-distribution. Unless the q's have Gaussian, and, hence, unbounded, distributions, (6.10) is altered for $\lambda \neq 0$, and the energy of the ensemble is not exactly conserved except at the end points $\lambda = 0$ and $\lambda = 1$. There are three alternatives open: (1) take a bounded q-distribution and live with the non-conservation in the hope that it will not be large, or will be insignificant in view of viscous damping; (2) take Gaussian q-distributions and hope, encouraged by the examples of § 3 and § 4, that the orthogonal expansion method may give good results anyhow; (3) put a strength factor in front of q_i in (6.5), or leave parameters free in the q-distribution, and make these quantities such functions of λ (to be developed in power series) that conservation in the mean is restored.

A degenerate form of (6.5)–(6.9), in which only current times appear and the q_i are suitably chosen white-noise processes, yields an expansion about the Markovian quasinormal approximation of Orszag (1970*b*) and shows that the latter has a model representation. This expansion may be regarded as intermediate between the strength-parameter expansion of § 5 and the present expansion about the direct-interaction approximation. It permits bounded y-distributions without complications involving conservation.

The concrete form taken by (6.5)–(6.9) for decaying isotropic turbulence is

$$(\partial/\partial t + \nu k^2)\, u_i(\mathbf{k},t) + (1-\lambda^2)\int_0^t \eta(k;t,s)\, u_i(\mathbf{k},s)\, ds$$
$$= (1-\lambda^2)^{\frac{1}{2}}\, q_i(\mathbf{k},t) - i\lambda k_l P_{ij}(\mathbf{k})\, \Sigma_{\mathbf{k}'}\, u_j(\mathbf{k}-\mathbf{k}',t)\, u_l(\mathbf{k}',t), \quad (6.12)$$

$$\eta(k;t,s) = \pi k \iint_\Delta b(k,p,q)\, G(p;t,s)\, U(q;t,s)\, pq\, dp\, dq, \quad (6.13)$$

$$q_i(\mathbf{k},t) = -ik_l P_{ij}(\mathbf{k})\, \Sigma_{\mathbf{k}'}\, \xi_j(\mathbf{k}-\mathbf{k}',t)\, \xi_l'(\mathbf{k}',t). \quad (6.14)$$

Here ν is kinematic viscosity, $u_i(\mathbf{k}, t)$ is a spatial Fourier component of the velocity field (subscripts are now vector indices), $\mathbf{k}$ is an allowed wave-vector in a large cyclic box, $P_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2)$, $G(p; t, s)$ is the average infinitesimal response function for wave-vector $p[G(p; t, t) = 1]$, $U(k; t, t')$ is the modal time covariance

$$\left[\langle \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}, t') \rangle = 4\pi \int_0^\infty U(k; t, t')\, k^2\, dk\right],$$

the integration Δ is over all p and q such that k, p, q form a triangle, and $b(k, p, q) = (p/k)(xy + z^3)$, where x, y, z are the interior-angle cosines opposite k, p, q, respectively. The solenoidal, isotropic vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are statistically independent of each other, and of the initial $\mathbf{u}$ distribution, and each have the same covariance tensor as $\mathbf{u}$.

These equations are of interest apart from their use in expanding about the direct-interaction approximation. At $\lambda = 0$, they give an amplitude-equation representation of the direct-interaction approximation which may facilitate numerical solution of the equations. Solution of (6.12) with an ensemble of $\mathbf{u}$ and $\boldsymbol{\xi}$ fields may be easier than solving the full integro-differential equations for $U(k; t, t')$ and $G(k; t, t')$ (Kraichnan 1964) because there are fewer time arguments. Equation (6.12) might also provide a useful improvement over simple eddy-viscosity formulas for representing the effects of subgrid scales in turbulence simulation computations. It provides for the possibility of back-flow of energy from the small scales, as well as eddy damping. Finally, the existence of the model equation (6.12) raises the possibility that model equations of similar type may exist that yield the Lagrangian-history direct-interaction approximation.

Note added in proof. Dr C. E. Leith (private communication) has independently proposed the new model equations (6.5)–(6.14) for $\lambda = 0$, in work which antedates that reported here.

7. Concluding remarks

From the point of view of basic turbulence theory, the principal import of the work reported above is, first, the conclusion that there is at least one way to construct approximations, for statistical functions, that are uniformly convergent in time, starting from a specification of the moments of the initial probability distribution. Of equal interest is the emergence of conditions which must be satisfied for this to be possible. One condition is the physically obvious requirement that the quantity averaged evolve without any singular behaviour in all the realizations of the ensemble. However, we also found that we could prove uniform convergence to the correct function, without ambiguity, only if the function in question was analytic at $t = 0$, a condition which could be assured only by excluding initial distributions that permitted unbounded amplitudes. Unbounded distributions, like the Gaussian, could be covered by the convergence theory only if they were represented as the limit of a sequence of bounded distributions. The implication is that the familiar diagram expansions, which result from averaging term-by-term over a Gaussian distribution, may actually not determine uniquely the functions they represent. Paradoxically, our approximation procedure gave excellent convergence when this caution was disregarded and, in §3 and §4, we treated series, with zero radius of convergence, obtained from Gaussian distributions. Moreover, some good results have been obtained for the diffusion problem of §4, with even less justification, by simply weighting the orders of the Gaussian irreducible-diagram expansion with powers of a formal parameter and then operating on the resulting series with the present orthogonal expansion method and with Padé approximants (Kraichnan 1968, 1970*a*).

The accuracy of the results obtained, in §4, from the first three terms of a series expansion of the Lagrangian velocity correlation in a non-trivial diffusion problem suggests that, apart from interest for basic theory, methods like the orthogonal expansion procedure may have great practical utility. The computer simulations with which the analytical results were compared required careful programming followed by several hours of computation time on an IBM 360-95 machine. In contrast, the evaluation of the Taylor series, and construction of the approximants, took much less time than did the simulation programming, and required no machine computation at all.

It is to be hoped that the convergence theory can be extended and that more efficient algorithms can be discovered and justified.

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